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Finite-Size Effects in Random Energy Models and in the Problem of Polymers in a Random Medium

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By the use of traveling wave equations we calculate the finite-size corrections to the free energy of random energy models in their low-temperature phases and in the neighborhood of the transition temperature. We find that although the extensive part of the free energy does not show any critical behavior when the temperature approaches its transition value, the finite-size corrections signal the transition by becoming singular. We obtain a scaling form for these finite-size corrections valid in the limit $N \to \infty$ and $T \to T_c$. By considering a generalized random energy model in the limit of a very large number of steps, we obtain results for the finite-size corrections in the problem of a polymer in a random medium.

KEY WORDS: Spin-glass; finite-size effects; directed polymers; traveling waves.

1. INTRODUCTION

There are several examples in the field of disordered systems in which phase transitions exist although there is no singular behavior in the extensive part of the free energy. For example, in the high-temperature phase of the Sherrington–Kirkpatrick model(1) the extensive part of the free energy is known and it does not possess any singularity at the phase transition. It is necessary to examine the finite-size corrections to the free energy in the high-temperature phase to observe singular behavior signaling the transition.(2) Other examples of disordered systems in which the branches

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of the free energy have no singularity at the phase transition include the random energy model (REM)(3–8) in its two phases, the generalized random energy model(9–16) in all its phases, and the problem of directed polymers in a random medium in the high-temperature phase(17,18) in all dimensions strictly larger than $2+1$. In all of these examples one expects the finite-size corrections to become singular at T_c, as a signal of the phase transition.(3,19)

In the present paper we calculate the finite-size corrections of the generalized random energy model (GREM) of an arbitrary number of steps p. We show that these finite-size corrections become singular at the phase transition and we obtain the scaling form of these corrections near the transition temperature T_c (in the limit $N \to \infty$ and $T \to T_c$). By taking the limit of a very large number of steps p in the GREM, we make the connection with the problem of directed polymers in the mean field case(20) and we recover the corrections expected from the analogy with traveling waves.(20,21) Lastly, we examine how the energy of a typical GREM configuration is distributed among its p steps. We obtain the result that, even for the GREM of a finite number of steps, the excess energy of the first steps of the GREM is comparable with what is expected for the directed polymer problem.(22)

The paper is arranged as follows. In Section 2 we define the GREM to be studied(10) and discuss how this model acts as an interpolation between the REM and the mean field limit of directed polymers in a random medium.(20) We point out the relationship between these problems and traveling wave equations, noticing that the finite-size corrections to the free energy of the GREM correspond to the dependence of the asymptotic behavior of a traveling wave equation upon the initial conditions.(21) The finite-size corrections of the free energy of the p-step GREM below the transition temperature are derived in Section 3. Section 4 contains the derivation of the corresponding results in the neighborhood of the transition temperature, yielding a scaling form in the limit $N \to \infty$ and $T \to T_c$. In Section 5 we consider the GREM when the number of steps p becomes very large. In this limit we recover the logarithmic corrections expected for the mean field directed polymer from the traveling wave approach. Finally, in Section 6 we calculate how the energy of a typical GREM configuration is distributed among its steps, using the expression for the finite-size corrections of Section 3.

The details of the calculations of Sections 3 and 4 require one to calculate the location of a front which is the solution of a traveling wave equation. This is discussed in Appendix A. The large-p behavior of two multiple integrals which we need in Sections 5 and 6 to treat the directed polymer problem is studied in Appendix B.

2. THE REM, GREM, AND THE MEAN FIELD THEORY OF POLYMERS IN A RANDOM MEDIUM

In this section we shall define the models which will be studied in the remainder of this paper, namely the REM, the GREM, and a mean field version of the problem of directed polymers in a random medium. We shall show how these problems can be written in the form of traveling wave equations$^{(20)}$ and discuss their relationships.

2.1. The p-Step GREM

2.1.1. Definition of the GREM. The generalized random energy model (GREM) was introduced$^{(9,10)}$ as a simplified spin-glass model for which the possible configurations of a system are organized according to an ultrametric structure (a tree structure). The model can be solved exactly for an arbitrary choice of the parameters $\{a_i\}$, $\{\alpha_i\}$ which define the model and so far it seems to be the only model for which it has been proved that the replica ansatz with a broken replica symmetry$^{(23,24)}$ gives the exact expression of the free energy.

In the GREM of p steps the possible configurations of the system are represented by the endpoints of a tree of p steps (see Fig. 1). The model is defined by two sets of numbers $\{\alpha_i > 1\}$, $\{a_i > 0\}$ for $1 \leqslant i \leqslant p$. We shall choose

$$\sum_{i=1}^{p} a_i = a \qquad \text{and} \qquad \prod_{i=1}^{p} \alpha_i = \alpha \tag{1}$$

The total number of branches at level i is $(\alpha_1 \alpha_2 \cdots \alpha_i)^N$. For each bond of the tree structure at level i, one chooses a random energy $\varepsilon_i^{(\mu)}$ according to

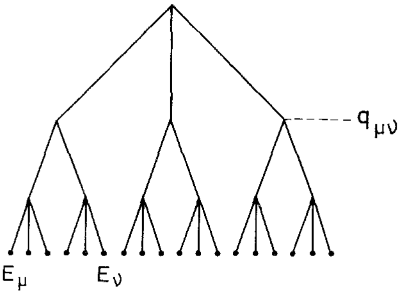

Fig. 1. The configurations of the GREM are the endpoints of a tree of p levels.

a probability distribution $P_i(\varepsilon)$. To keep the discussion simple, we shall take this to be

$$P_i(\varepsilon) = (N\pi a_i)^{-1/2} \exp(-\varepsilon^2/Na_i) \tag{2}$$

By definition of the model, the configurations are identified with the endpoints of the tree, and the energy E_μ of a configuration μ is given by

$$E_\mu = \sum_{i=1}^{p} \varepsilon_i^{(\mu)} \tag{3}$$

where the sum runs over all bonds connecting the configuration μ to the top of the tree. The partition function Z_p for a GREM of p steps is then defined as

$$Z_p = \sum_\mu \exp(-E_\mu/T) \tag{4}$$

where T is the temperature.

For $p = 1$, the model reduces to the random energy model (REM) in which there are no correlations between the energies of different configurations. Therefore the system consists of α^N configurations. The energy E_μ of each configuration μ is chosen at random according to a distribution $\mathscr{P}(E)$,

$$\mathscr{P}(E) = (\pi Na)^{-1/2} \exp(-E^2/Na) \tag{5}$$

and the energies E_μ and E_ν of two different configurations are uncorrelated,

$$\mathscr{P}_{\mu,\nu}(E_\mu, E_\nu) = \frac{1}{\pi Na} \exp\left(-\frac{E_\mu^2 + E_\nu^2}{Na}\right) \tag{6}$$

For $p > 1$, the distribution of the energy of each individual configuration is still given by (5), but the energies of the configurations are correlated. Two configurations μ and ν which belong to the same branch up to level $i-1$ and then bifurcate at level i have two energies E_μ and E_ν which are correlated,

$$\mathscr{P}_{\mu,\nu}(E_\mu, E_\nu) = \frac{1}{\pi Na(1-q^2)^{1/2}} \exp\left[-\frac{E_\mu^2 + E_\nu^2 - 2qE_\mu E_\nu}{Na(1-q^2)}\right] \tag{7}$$

where

$$q = q_{\mu,\nu} = \frac{1}{a} \sum_{j=1}^{i-1} a_j \tag{8}$$

So, the correlation between the energies of the two configurations depends on a parameter q which measures their distance on the tree. This parameter q can be thought of as the overlap between the two configurations.

The number of configurations ν which have an overlap $q_{\mu,\nu}$ given by (8) with a given configuration μ is

$$(\alpha_i \alpha_{i+1} \cdots \alpha_p)^N - 1 \tag{9}$$

2.1.2. The Exact Expression of the Free Energy.

It will be convenient to define a set of parameters A_i,

$$A_i = 2\left(\frac{\log \alpha_i}{a_i}\right)^{1/2} \tag{10}$$

The GREM can be solved exactly for an arbitrary choice of the $\{\alpha_i\}$ and of the $\{a_i\}$. Here, for simplicity, we shall only consider cases of the GREM in which $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_p$. In this case it has been shown(10) that for $T \leqslant 1/A_p$ one has a low-temperature frozen phase in which

$$\lim_{N \to \infty} T \frac{\langle \log Z_p \rangle}{N} = \sum_{i=1}^{p} (a_i \log \alpha_i)^{1/2} \tag{11}$$

and above a temperature $1/A_1$, one has a high-temperature phase where

$$\lim_{N \to \infty} T \frac{\langle \log Z_p \rangle}{N} = T \log \alpha + \frac{a}{4T} \tag{12}$$

We use $\langle \cdots \rangle$ to denote an average over disorder. If $A_1 = A_2 = \cdots = A_p$, then these are the only two phases that exist. However, in general, for $A_1 < A_2 < \cdots < A_p$, one finds p transition temperatures $1/A_i$ and one has $p-1$ intermediate phases.

As a result of the tree structure of the GREM, one can write a recursion relation involving the partition function Z_p of a GREM of p steps, with $\{\alpha_i\} = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and $\{a_i\} = \{a_1, a_2, \ldots, a_p\}$, and the partition function Z_{p-1} of a $(p-1)$-step GREM, with $\{\alpha_i\} = \{\alpha_2, \alpha_3, \ldots, \alpha_p\}$ and $\{a_i\} = \{a_2, a_3, \ldots, a_p\}$,

$$Z_p = \sum_{i=1}^{\alpha_1^N} Z_{p-1}^{(i)} \exp(-\varepsilon_i/T) \tag{13}$$

where ε_i is the energy of the ith first-step bond and the $Z_{p-1}^{(i)}$ are α_1^N independent $(p-1)$-step GREM partition functions. This follows because all configurations of the p-step GREM must use one of the α_1^N first-step bonds and can then follow any configuration on one of the remaining α_1^N $(p-1)$-step GREMs.

2.1.3. The Traveling Wave Approach. The partition function Z_p is a function of all the energies $\varepsilon_i^{(\mu)}$ of the tree. So it is a random variable and the meaningful object to consider is its probability distribution or a generating function. It turns out that one can define a generating function $G_p(x)$ by

$$G_p(x) = \langle \exp(-Z_p e^{-x/T}) \rangle \tag{14}$$

which satisfies a very simple recursion relation

$$G_p(x) = \left[\int \frac{d\varepsilon}{(N\pi a_1)^{1/2}} G_{p-1}(x+\varepsilon) \exp\left(\frac{-\varepsilon^2}{Na_1}\right) \right]^{\alpha_1^N} \tag{15}$$

Equation (15) is obtained by replacing Z_p by its expression (13) in $G_p(x)$ and by using the fact that the ε_i and the $Z_{p-1}^{(i)}$ are independent random variables. Notice that the temperature T does not appear in the recursion (15). So the whole temperature dependence comes from the initial condition

$$G_0(x) = \exp(-e^{-x/T}) \tag{16}$$

which follows from the fact that $Z_0 = 1$.

Equation (15) is a traveling wave equation, very similar to the equation that we shall derive for the mean field directed polymer problem.(20) It reduces the calculation of any thermodynamic quantity to a finite number of iterations (p times) of the traveling wave equation (15) with the initial condition (16).

2.2. The Mean Field Directed Polymer Problem

2.2.1. Definition. A mean field version of the problem of directed polymers in a random medium can be obtained by formulating the problem on a branch of a tree (see Fig. 2).(20) One has a tree structure with coordination number $K+1$ ($K=2$ in Fig. 2) and to each bond b of the tree one assigns a random energy ε_b according to a given probability distribution $\rho(\varepsilon)$. One then considers all directed polymers, or walks, of length L emanating from the root 0 and running down the tree. One such walk is shown in bold in Fig. 2. The energy E_W of such a walk W is defined to be the sum of the energies on the bonds visited by W,

$$E_W = \sum_{b \in W} \varepsilon_b \tag{17}$$

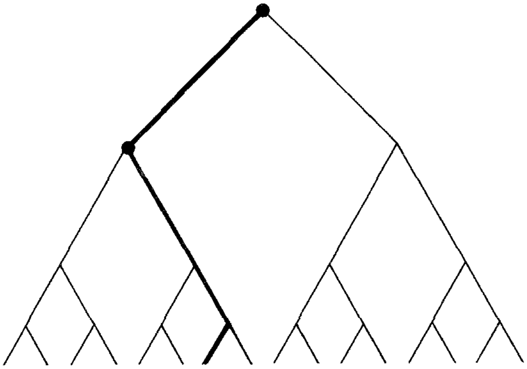

Fig. 2. A configuration of a directed polymer on a tree ($K=2$).

and the partition function Z_L is then

$$Z_L = \sum_W \exp(-E_W/T) \tag{18}$$

where the sum runs over all directed walks of length L.

For a Gaussian distribution of the bond energies

$$\rho(\varepsilon) = \frac{1}{\sqrt{\pi}} \exp(-\varepsilon^2) \tag{19}$$

the system undergoes a phase transition at a temperature

$$T_c = \frac{1}{2(\log K)^{1/2}} \tag{20}$$

and the free energies of the low- and the high-temperature phases are given by

$$\lim_{L\to\infty} -T\frac{\langle \log Z_L \rangle}{L} = -(\log K)^{1/2} \qquad \text{for} \quad T < T_c \tag{21}$$

$$\lim_{L\to\infty} -T\frac{\langle \log Z_L \rangle}{L} = -T \log K - \frac{1}{4T} \qquad \text{for} \quad T > T_c \tag{22}$$

2.2.2. Analogy with the GREM. By comparing (11) and (12) with (21) and (22), we see that the expression for the free energy is very similar in the directed polymer problem and in the GREM. The analogy between the two problems can best be understood by considering the correlations between the energies of two configurations.

For the directed polymer problem the probability distribution $\mathscr{P}(E)$ of the energy E_W of a walk W of length L is given, when $\rho(\varepsilon)$ is a Gaussian (19), by

$$\mathscr{P}(E) = (\pi L)^{-1/2} \exp(-E^2/L) \tag{23}$$

whereas the probability $\mathscr{P}_{W,W'}(E_W, E_{W'})$ that the two walks W and W' have energies E_W and $E_{W'}$ is given by

$$\mathscr{P}_{W,W'}(E_W, E_{W'}) = \frac{1}{\pi L(1-q^2)^{1/2}} \exp\left(-\frac{E_W^2 + E_{W'}^2 - 2qE_W E_{W'}}{L(1-q^2)}\right) \tag{24}$$

where q is the fraction of their length that the walks W and W' have in common.

The number of walks W' which have an overlap q with a given walk W is

$$K^{L(1-q)} - 1 \tag{25}$$

We see, from (5), (7), (9), and (23)–(25), that the correlations between the energies of the configurations in the GREM and in the directed polymer problem have very similar expressions. The only difference is that the overlap q is continuous in the directed polymer problem, whereas it is discrete in the GREM. If one considers a GREM where all the a_i and all the α_i are equal,

$$a = \frac{L}{N}; \qquad a_i = \frac{L}{pN}; \qquad \log \alpha_i = \frac{L}{pN} \log K \tag{26}$$

one sees that for $p = L$, the GREM and the directed polymer problem become identical. So by varying p one can interpolate between the REM and the directed polymer problem.

2.2.3. Traveling Wave Approach. Let us now consider how the mean field directed polymer problem can be cast in the form of a traveling wave equation. At the first step all the walks of length $L+1$ must pass through one of the K bonds $b_1, b_2, \ldots, b_K$ (see Fig. 2). The walks can then take any directed path of length L on one of the K independent branches with roots at the points i. This gives a recursion for Z_L:

$$Z_{L+1} = \sum_{i=1}^{K} Z_L^{(i)} \exp(-\varepsilon_i/T) \tag{27}$$

where ε_i is the energy associated with the bond b_i and $Z_0 = 1$. This recursion is analogous to (13) for the GREM. So, as before, it is convenient to introduce the generating function $G_L(x)$,

$$G_L(x) = \langle \exp(-Z_L e^{-x/T}) \rangle \tag{28}$$

In terms of this generating function, (27) becomes[(20)]

$$G_{L+1}(x) = \left[\int d\varepsilon\, \rho(\varepsilon)\, G_L(x+\varepsilon) \right]^K \tag{29}$$

with the initial condition

$$G_0(x) = \exp(-e^{-x/T}) \tag{30}$$

So this problem can be reduced to a traveling wave equation very similar to that for the GREM [compare (29) and (30) with (15) and (16)].

2.2.4. Analogy with Reaction-Diffusion Equations. In the limit of large L, the solutions of (29) become traveling waves of the form[(20)]

$$G_L(x) = W(x - cL + d(L)) \tag{31}$$

where $d(L)/L \to 0$ as $L \to \infty$. The velocity of the wavefront c depends on the initial condition (30). Above the transition temperature T_c, the velocity $c(T)$ is given by

$$c(T) = T \log\left[K \int d\varepsilon\, \rho(\varepsilon)\, e^{-\varepsilon/T} \right] \qquad \text{for} \quad T > T_c \tag{32}$$

whereas below T_c the velocity is given by the minimal velocity

$$c(T) = T_c \log\left[K \int d\varepsilon\, \rho(\varepsilon)\, e^{-\varepsilon/T_c} \right] \qquad \text{for} \quad T \leqslant T_c \tag{33}$$

where T_c is the solution of

$$\frac{d}{dT} T \log\left[K \int d\varepsilon\, \rho(\varepsilon)\, e^{-\varepsilon/T} \right] \Bigg|_{T = T_c} = 0 \tag{34}$$

From the definition of the generating function (28), one can see that this velocity $c(T)$ corresponds to the limit of $T\langle \log Z_L \rangle / L$ for large L in the polymer problem, since the values of x where $G_p(x)$ is noticeably different from 0 and 1 are of order of $T \log Z_{\text{typical}}$. For a Gaussian choice of $\rho(\varepsilon)$ one recovers the expressions for T_c and the free energy, (20)–(22).

It has been observed[(20)] that the discrete-time traveling wave equation

(29) with the initial condition (30) is closely related to the Kolmogorov–Petrovsky–Piscounov (KPP) equation,

$$\frac{\partial g(x,t)}{\partial t}=\frac{1}{2}\frac{\partial^2 g(x,t)}{\partial x^2}+g^2(x,t)-g(x,t) \tag{35}$$

The KPP equation also admits traveling wave solutions and the approach of the system toward these asymptotic solutions has been studied by Bramson.[(21)] If the initial condition $g(x,0)$ is an increasing function of x with $g(x,0)\to 0$ as $x\to -\infty$ and $g(x,0)\to 1$ as $x\to +\infty$, then the velocity c_β and the shape of the traveling wave w_β depend on the asymptotic shape of the initial condition for $x\to\infty$. If the initial condition $g(x,0)$ is characterized by an exponential decay

$$1-g(x,0)\approx e^{-\beta x} \qquad \text{for} \quad x\to\infty \tag{36}$$

then Bramson was able to show that[(21,25,26)]

$$g(x,t)=w_\beta(x-m_\beta(t)) \tag{37}$$

where for large t

$$m_\beta(t)=\frac{1}{2}\left(\beta+\frac{1}{\beta}\right)t+O(1) \qquad \text{if} \quad \beta<\sqrt{2} \tag{38}$$

$$=\sqrt{2}\,t-2^{-3/2}\log t+O(1) \qquad \text{if} \quad \beta=\sqrt{2} \tag{39}$$

$$=\sqrt{2}\,t-3.2^{-3/2}\log t+O(1) \qquad \text{if} \quad \beta>\sqrt{2} \tag{40}$$

By analogy, one might expect the traveling wave equation (29) with initial condition (30) to have similar corrections. As the traveling wave velocity c corresponds to $T<\log Z_L>/L$ for the polymer, one might therefore suppose that the finite-size corrections for the polymer will be of the form

$$\langle\log Z_L\rangle=L\log\left[K\int d\varepsilon\,\rho(\varepsilon)\,e^{-\varepsilon/T}\right]+O(1), \qquad T>T_c \tag{41}$$

$$=L\log\left[K\int d\varepsilon\,\rho(\varepsilon)\,e^{-\varepsilon/T_c}\right]-\frac{1}{2}\log L+O(1), \qquad T=T_c \tag{42}$$

$$=\frac{T_c}{T}L\log\left[K\int d\varepsilon\,\rho(\varepsilon)\,e^{-\varepsilon/T_c}\right]-\frac{3T_c}{2T}\log L+O(1), \quad T<T_c \tag{43}$$

where T_c is the solution of (34). The extensive terms in (41)–(43) were obtained in ref. 20. In this paper we shall obtain these finite-size corrections, by calculating the finite-size corrections for the GREM for an arbitrary number of steps p and then taking the limit of large p.

3. FINITE-SIZE CORRECTIONS TO THE FREE ENERGY OF THE p-STEP GREM IN THE LOW-TEMPERATURE PHASE

We shall now derive the finite-size corrections to $\langle \log Z_p \rangle$ for the low-temperature phase of the p-step GREM up to terms of order one. To do this we shall use the recursion (15), obtained for the generating function $G_p(x)$, defined in (14). As we are only interested in calculating corrections up to order one, we shall replace (15) by

$$G_p(x) = \exp\left\{ -(N\pi a_1)^{-1/2}\, \alpha_1^N \int d\varepsilon [1 - G_{p-1}(x+\varepsilon)] \exp(-\varepsilon^2/Na_1) \right\} \tag{44}$$

This simplified equation can be understood by noticing that the only range of values x where $G_p(x)$ is not exponentially small in N is when the integral in the bracket on the rhs of (15) is very close to one. So, in that range of values of x, one can replace (15) by (44) by neglecting exponentially small factors.

By iterating (44) p times one can obtain the generating function for a p-step GREM, $G_p(x)$, in terms of the initial condition $G_0(x)$ given by (16). Once one knows $G_p(x)$ one can find the average free energy using an integral representation. If one makes the substitution

$$t = \exp(-x/T) \tag{45}$$

one can use the integral representation of $\langle \log Z \rangle$,

$$\langle \log Z \rangle = \int_0^\infty dt \left(\frac{e^{-t} - \langle e^{-tZ} \rangle}{t} \right) \tag{46}$$

to show that

$$\langle \log Z \rangle = \int_0^\infty dt \left(\frac{e^{-t} - G_p(-T \log t)}{t} \right) \tag{47}$$

If $T = 0$, it is simpler to extract the desired information from the generating function, as one can then obtain the probability distribution of the ground-state energy $\Pi_{\mathrm{GS}}(E)$ by

$$\Pi_{\mathrm{GS}}(E) = \left. \frac{dG_p(x)}{dx} \right|_{x=E} \tag{48}$$

The iteration of (15), or even of its simplified version (44), is not easy to do. The expression for $G_p(x)$ becomes more and more complicated as p increases. However, by using the fact that N is large, one can greatly simplify the calculation and iterate $G_p(x)$ an arbitrary number of times. The details of the calculations which lead to the expression for G_p are presented in Appendix A. Here, let us just discuss the results.

As already discussed in Section 2, the free energy of the GREM depends on the choice of the parameters $\{a_i\}$ and $\{\alpha_i\}$. Here we shall limit our discussion to two cases [see (10) for the definition of the A_i]: the first case for which the freezing at each level of the GREM occurs at a different temperature (the gradual freezing $A_1 < A_2 < \cdots < A_p$) and the second case for which all the levels of the GREM freeze at the same temperature (the simultaneous freezing $A_1 \simeq A_2 \simeq \cdots \simeq A_p$).

3.1. The Gradual Freezing case: $A_1 < A_2 < \cdots < A_p < 1/T$

In this case the system undergoes p phase transitions and the results we are going to discuss are valid in the low-temperature phase. One can show (see Appendix A) that in the range of values of x such that

$$x = \sum_{i=1}^{p} N(a_i \log \alpha_i)^{1/2} - \sum_{i=1}^{p} \frac{1}{2A_i} \log(4N\pi \log \alpha_i) + \frac{1}{A_p} \log \Gamma(1 - TA_p) + \sum_{i=1}^{p-1} \frac{1}{A_i} \log \Gamma\left(1 - \frac{A_i}{A_{i+1}}\right) + y \tag{49}$$

where y is of order one, $G_p(x)$ is given by

$$G_p(x) = \exp[-\exp(-A_1 y)] \tag{50}$$

So, in the region where $G_p(x)$ is noticeably different from zero or one, $G_p(x)$ always has the same form for all p, up to a translation, which is given by (49).

From (47), (49), and (50), one can easily get an expression for $\langle \log Z_p \rangle$:

$$\langle \log Z_p \rangle = \frac{N}{T} \sum_{i=1}^{p} (a_i \log \alpha_i)^{1/2} - \frac{1}{2T} \sum_{i=1}^{p} \frac{\log(4N\pi \log \alpha_i)}{A_i} + \frac{1}{TA_p} \log[\Gamma(1 - A_p T)] + \frac{1}{T} \sum_{i=1}^{p-1} \frac{1}{A_i} \log\left[\Gamma\left(1 - \frac{A_i}{A_{i+1}}\right)\right] + \left(1 - \frac{1}{TA_1}\right) \Gamma'(1) \tag{51}$$

So the leading finite-size correction$^{(14)}$ to $\log Z$ is of order $-\log N$. We see that as $T \to A_p^{-1}$ (the lowest transition temperature), the third term in (51) signals the transition by becoming singular. Also, we see that if one tries to relax the condition $A_1 < A_2 < \cdots < A_p$ by making $A_{i+1} - A_i$ small, the term of order one diverges.

When $p = 1$ one can check that (51) reduces to

$$\langle \log Z_1 \rangle = \frac{N(a \log \alpha)^{1/2}}{T} - \frac{T_c}{2T} \log(4N\pi \log \alpha) + \left(1 - \frac{T_c}{T}\right) \Gamma'(1) + \frac{T_c}{T} \log \Gamma\left(1 - \frac{T}{T_c}\right) \tag{52}$$

where we have used the fact that

$$T_c = \frac{1}{2}\left(\frac{a}{\log \alpha}\right)^{1/2} \tag{53}$$

for the REM. This is exactly the result obtained in ref. 3 (in the case $a = 1$ and $\alpha = 2$).

The explicit form (50) allows one to obtain other information about the distribution of $\log Z$. For example, the variance of the free energy is

$$\begin{aligned} \langle (\log Z_p)^2 \rangle - \langle \log Z_p \rangle^2 &= \left(\frac{1}{(A_1 T)^2} - 1\right) [\Gamma''(1) - \Gamma'(1)^2] \\ &= \left(\frac{1}{(A_1 T)^2} - 1\right) \frac{\pi^2}{12} \end{aligned} \tag{54}$$

for the p-step GREM, whereas in the REM (i.e., $p = 1$), it becomes

$$\langle (\log Z_1)^2 \rangle - \langle \log Z_1 \rangle^2 = \left(\frac{T_c^2}{T^2} - 1\right) \frac{\pi^2}{12} \tag{55}$$

3.2. The Simultaneous Freezing Case: $A_1 \simeq A_2 \simeq \cdots \simeq A_p < 1/T$

In the last section we have seen that if all the A_i become roughly equal, the expression (51) for the finite-size correction to the free energy becomes singular. As explained in Appendix A, if

$$A_i = A + z_i N^{-1/2} \tag{56}$$

where each z_i is of order one, $G_p(x)$ is modified and becomes

$$G_p(x) = \exp[-\exp(-Ay)] \tag{57}$$

where y is of order one and is now defined by

$$x=\sum_{i=1}^{p} N(a_i \log \alpha_i)^{1/2}-\frac{1}{2A}\log(4\pi N\log\alpha_p)+\frac{1}{A}\log\Gamma(1-TA)$$
$$+\frac{1}{A}\log J_1(p-1)+y \tag{58}$$

where the integral $J_1(p-1)$ is given by

$$J_1(p-1)=\int_0^\infty\frac{d\psi_1}{(\pi a_1)^{1/2}}\int_0^\infty\frac{d\psi_2}{(\pi a_2)^{1/2}}\cdots\int_0^\infty\frac{d\psi_{p-2}}{(\pi a_{p-2})^{1/2}}$$
$$\times\int_0^\infty\frac{d\psi_{p-1}}{(\pi a_{p-1})^{1/2}}\exp\left[-\frac{\psi_1^2}{a_1}-\frac{(\psi_2-\psi_1)^2}{a_2}\right.$$
$$-\frac{(\psi_3-\psi_2)^2}{a_3}\cdots-\frac{(\psi_{p-1}-\psi_{p-2})^2}{a_{p-1}}-\frac{\psi_{p-1}^2}{a_p}$$
$$\left.-(z_2-z_1)\psi_1-(z_3-z_2)\psi_2\cdots-(z_p-z_{p-1})\psi_{p-1}\right] \tag{59}$$

So, as in Section 3.1, the shape of $G_p(x)$ is given by the simple expression (57), independent of p up to a translation given by (58). Then $\langle\log Z_p\rangle$ can be found using the integral representation of (47), to give, correct to order one,

$$\langle\log Z_p\rangle=\frac{N}{T}\sum_{i=1}^{p}(a_i\log\alpha_i)^{1/2}-\frac{1}{2AT}\log(4\pi N\log\alpha_p)$$
$$+\frac{1}{AT}\log\Gamma(1-AT)+\frac{1}{AT}\log J_1(p-1)+\left(1-\frac{1}{AT}\right)\Gamma'(1) \tag{60}$$

Since $T_c\simeq 1/A$, we see that the leading correction is $-(T_c/2T)\log N$, identical to that for the random energy model (52). As the shape of $G_p(x)$ is always the same, one expects the fluctuations to keep the same form (55) as before. The case where the z_i are all equal will be of particular interest, because it is this structure that one expects to yield the answer for the mean field directed polymer problem as $p\to\infty$. This limit of (60) will be considered in Section 5.

4. FINITE-SIZE CORRECTIONS TO THE FREE ENERGY OF THE p-STEP GREM NEAR T_c: THE SCALING FORM

In this section we shall obtain, correct to terms of order one, the finite-size corrections to $\langle \log Z_p \rangle$ for the p-step GREM in the neighborhood of the phase transition to the high-temperature phase. We shall therefore be interested in cases where $|A_p - 1/T|$ is small. As in the previous section, the finite-size corrections will be derived by iterating the recursion (44) p times and then using the integral relation (47). We shall consider two cases separately as in Section 3: the gradual freezing case $A_1 < A_2 < \cdots < A_p \simeq 1/T$ and the simultaneous freezing case $A_1 \simeq A_2 \simeq \cdots \simeq A_p \simeq 1/T$.

4.1. The gradual freezing case: $A_1 < A_2 < \cdots < A_p \simeq 1/T$

Here one is interested in a p-step GREM with p transition temperatures, $T_1 > T_2 > \cdots > T_p$, with the temperature in the neighborhood of the lowest of these transitions. One finds (see Appendix A) that $G_p(x)$ still has the form of (50), but that now

$$x = \sum_{i=1}^{p} N(a_i \log \alpha_i)^{1/2} - \sum_{i=1}^{p-1} \frac{1}{2A_i} \log(4N\pi \log \alpha_i)$$
$$+ \sum_{i=1}^{p-1} \frac{1}{A_i} \log \Gamma\left(1 - \frac{A_i}{A_{i+1}}\right) + \frac{1}{A_p}$$
$$\times \log\left[\frac{1}{2} \exp\left\{\frac{Na_p}{4}\left(\frac{1}{T} - A_p\right)^2\right\} \operatorname{erfc}\left\{\frac{(Na_p)^{1/2}(1/T - A_p)}{2}\right\}\right] + y \qquad (61)$$

where erfc (x) is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-y^2)\, dy \qquad (62)$$

One sees that (61) is very to similar the translation of (49) in the previous section, the terms resulting from iterating the recursion over steps $p-1$ to 1 remaining the same and only the term from the recursion from step p to $p-1$ being altered. It is simple to check that the limit $\sqrt{N}\,(1/T - A_p) \to \infty$ of (61) is identical to the limit $T \to A_p^{-1}$ of (49), so that the two expressions for $G_p(x)$ below and near the transition are consistent with each other. Using the integral representation of (47), one can now show, from (61), that

$$\begin{aligned}\langle \log Z_p \rangle = {} & \frac{N}{T} \sum_{i=1}^{p} (a_i \log \alpha_i)^{1/2} - \frac{1}{2T} \sum_{i=1}^{p-1} \frac{\log(4N\pi \log \alpha_i)}{A_i} \\ & + \frac{1}{T} \sum_{i=1}^{p-1} \frac{1}{A_i} \log \Gamma\left(1 - \frac{A_i}{A_{i+1}}\right) + \left(1 - \frac{1}{TA_1}\right) \Gamma'(1) \\ & + \frac{1}{A_p T} \log \left[\frac{1}{2} \exp\left\{ \frac{Na_p}{4} \left(\frac{1}{T} - A_p \right)^2 \right\} \right. \\ & \left. \times \operatorname{erfc}\left\{ \frac{(Na_p)^{1/2}\,(1/T - A_p)}{2} \right\} \right] \end{aligned} \tag{63}$$

The leading correction to $\langle \log Z \rangle$ remains of order $-\log N$ at, and in the neighborhood of, the transition when $p > 1$. One can see that the scaling function [the last term in (63)] is a function of $(Na_p)^{1/2}\,(1/T - A_p)$. One therefore has to compare the deviation from the transition temperature $1/A_p$ with $N^{-1/2}$ to determine the behavior of the finite-size corrections to the free energy. One can check that, in the limit $\sqrt{N}\,(1/T - A_p) \to \infty$, the scaling functions agrees with the result of (51) in the limit $1/T - A_p \to 0$.

The result for the REM can be obtained by setting $p = 1$ in (63) and by using the expression (53) for T_c. Doing this, one obtains

$$\begin{aligned}\langle \log Z_1 \rangle = {} & \frac{N}{T} (a \log \alpha)^{1/2} \\ & + \frac{T_c}{T} \log \left[\frac{1}{2} \exp\left\{ \frac{N}{4} \left(\frac{1}{T} - \frac{1}{T_c} \right)^2 \right\} \operatorname{erfc}\left\{ \frac{\sqrt{N}}{2} \left(\frac{1}{T} - \frac{1}{T_c} \right) \right\} \right] \end{aligned} \tag{64}$$

[the term $\Gamma'(1)$ is not present because it would be of order $1/\sqrt{N}$]. We see that for the REM, in the neighborhood of T_c, if $T - T_c \simeq N^{-1/2}$, the finite-size correction to the free energy is no longer proportional to $\log N$, but becomes of order 1. One can also show that for the REM, (64) interpolates between the known results for $T < T_c$ and $\sqrt{3}\, T_c > T > T_c$ in ref. 3.

4.2. The Simultaneous Freezing Case: $A_1 \simeq A_2 \simeq \cdots \simeq A_p \simeq 1/T$

Now we shall turn to the case that will be of interest to us later when we wish to obtain results valid for the mean field directed polymer. We have

$$A_i = A + z_i N^{-1/2} \qquad \text{with } z_i = O(1) \text{ for } 1 \leqslant i \leqslant p \tag{65}$$

$$1/T = A + z_{p+1} N^{-1/2} \qquad \text{with } z_{p+1} = O(1) \tag{66}$$

As in the case of Section 4.1, the fact that $(1/T-A)$ is small means that the behavior related to carrying out the recursion from step p of the GREM to step $p-1$ is altered. Using the methods discussed in Appendix A, one now obtains the result that

$$G_p(x)=\exp[-\exp(-Ay)] \tag{67}$$

where now y is related to x by

$$x=\sum_{i=1}^{p} N(a_i \log \alpha_i)^{1/2}+\frac{1}{A}\log J_2(p)+y \tag{68}$$

with

$$\begin{aligned} J_2(p)=&\int_0^\infty \frac{d\psi_1}{(\pi a_1)^{1/2}}\int_0^\infty \frac{d\psi_2}{(\pi a_2)^{1/2}}\cdots\int_0^\infty \frac{d\psi_p}{(\pi a_p)^{1/2}} \\ &\times\exp\left[-\frac{(\psi_p-\psi_{p-1})^2}{a_p}-\frac{(\psi_{p-1}-\psi_{p-2})^2}{a_{p-1}}\cdots-\frac{(\psi_2-\psi_1)^2}{a_2}\right. \\ &\left.-\psi_1^2/a_1-(z_{p+1}-z_p)\,\psi_p-(z_p-z_{p-1})\,\psi_{p-1}\cdots-(z_2-z_1)\,\psi_1\right] \end{aligned} \tag{69}$$

Using the integral representation of (47), one accordingly finds that

$$\langle \log Z_p\rangle=\frac{N}{T}\sum_{i=1}^{p}(a_i\log\alpha_i)^{1/2}+\log J_2(p) \tag{70}$$

One must now examine the behavior of $J_2(p)$ to determine the leading correction to $\langle \log Z_p\rangle$. The above expression again provides one with a scaling form for the finite-size corrections to the free energy. One finds that the scaling function is a function of $\sqrt{N}\,(1/T-A_p)$. Again one can check that the limit $(z_{p+1}-z_p)\sqrt{N}\to\infty$ of (70) gives the same result as the limit $T\to A^{-1}$ in (60). We shall return to a consideration of these finite-size corrections when p is large in the next section.

5. FINITE-SIZE CORRECTIONS FOR DIRECTED POLYMERS IN A RANDOM MEDIUM

In Section 2 we discussed the relationship between the p-step GREM and the mean field limit of directed polymers in a random medium. If one increases the number p of steps of the GREM, keeping the branching ratios and the energy distributions given by (26), the correlations between energies of different configurations become more and more like those of the mean field directed polymer problem. In this section, therefore, we shall consider the results obtained in the previous two sections in the limit that p becomes large (in particular $p=L$) with the a_i and α_i given by (26)

$$a=\frac{L}{N}; \qquad a_i=\frac{L}{pN}; \qquad \log \alpha_i=\frac{L}{pN}\log K$$

to obtain answers that should be valid for the mean field directed polymer problem.

5.1. The Low-Temperature Phase: $T<T_c$

In the low-temperature phase, we can use the results obtained for the GREM in Section 3.2. Since all the a_i and α_i are equal [see (26)], the A_i are given by

$$A_i=A=\frac{1}{2(\log K)^{1/2}}=\frac{1}{T_c} \tag{71}$$

and the results (59)–(60) have to be used with the a_i and α_i given by (26) and in the case where all the $z_i=0$. Therefore

$$\begin{aligned}\langle \log Z_p \rangle &= \frac{L}{T}(\log K)^{1/2}-\frac{T_c}{2T}\log\left(4\pi\frac{L}{p}\log K\right)+\frac{T_c}{T}\log\Gamma\left(1-\frac{T}{T_c}\right)\\ &\quad +\frac{T_c}{T}\log\tilde{J}_1(p-1)+\left(1-\frac{T_c}{T}\right)\Gamma'(1)\end{aligned} \tag{72}$$

where the integral $\tilde{J}_1(p-1)$ given by (59) becomes [all $z_i=0$ because of (71)]

$$\begin{aligned}\tilde{J}_1(p-1)=\int_0^\infty\frac{d\psi_1}{\sqrt{\pi}}\int_0^\infty\frac{d\psi_2}{\sqrt{\pi}}\cdots\int_0^\infty\frac{d\psi_{p-1}}{\sqrt{\pi}}\\ \times\exp[-\psi_{p-1}^2-(\psi_{p-1}-\psi_{p-2})^2\cdots-(\psi_2-\psi_1)^2-\psi_1^2]\end{aligned} \tag{73}$$

This integral is discussed in Appendix B. It can be related to the probability that the displacement of a Brownian particle, starting at 0 at time 0 and returning to 0 at time p, remains positive at all positive integer times ($1\leqslant t\leqslant p-1$). One can show (Appendix B) that for large p the integral $\tilde{J}_1(p-1)$ behaves as

$$\tilde{J}_1(p-1)\simeq\frac{C_1}{p^{3/2}} \tag{74}$$

where C_1 lies between two bounds $(0.318 < p^{3/2}\tilde{J}_1(p-1) < 2.36)$ and appears numerically to tend to a limit

$$C_1 = 1.00 \pm 0.05 \tag{75}$$

as p becomes large. Substituting (74) in (72) for large p, one sees that for $T < T_c$,

$$\begin{aligned}\langle \log Z_p \rangle = \frac{L}{T}(\log K)^{1/2} - \frac{T_c}{2T}\log\left(\frac{L}{p}\right) - \frac{3T_c}{2T}\log p + \frac{T_c}{T}\log C_1 \\ - \frac{T_c}{2T}\log(4\pi \log K) + \left(1 - \frac{T_c}{T}\right)\Gamma'(1) + \frac{T_c}{T}\log\Gamma\left(1 - \frac{T}{T_c}\right)\end{aligned} \tag{76}$$

We see that as long as $L \to \infty$ with fixed p, the finite-size correction is $-(T_c/2T)\log L$ as in the REM (52). On the contrary, if $p \to \infty$ when $L \to \infty$, keeping the ratio L/p fixed (for example, $L/p = 1$ as discussed in Section 2) we get a finite-size correction $-(3T_c/2T)\log L$ as in the case of the KPP equation (40) and (43).

5.2. The Transition Temperature: $T = T_c$

To obtain the finite-size corrections at $T = T_c$, we can use the results of Section 4.2. The A_i are still given by (71) (implying that all the $z_i = 0$ for $1 \leqslant i \leqslant p$) and since we are considering the case $T = T_c$ one also has $z_{p+1} = 0$. Therefore the results of Section 4.2 become

$$\langle \log Z_p \rangle = \frac{L}{T}(\log K)^{1/2} + \log \tilde{J}_2(p) \tag{77}$$

where $\tilde{J}_2(p)$ is given by

$$\begin{aligned}\tilde{J}_2(p) = \int_0^\infty \frac{d\psi_1}{\sqrt{\pi}} \int_0^\infty \frac{d\psi_2}{\sqrt{\pi}} \cdots \int_0^\infty \frac{d\psi_p}{\sqrt{\pi}} \\ \times \exp[-(\psi_p - \psi_{p-1})^2 \cdots - (\psi_2 - \psi_1)^2 - \psi_1^2]\end{aligned} \tag{78}$$

The integral $\tilde{J}_2(p)$ is discussed in Appendix B, where it is shown that for large p

$$\tilde{J}_2(p) \simeq \frac{C_2}{p^{1/2}} \tag{79}$$

As for the integral $\tilde{J}_1(p-1)$, one can obtain bounds $(0.318 < p^{1/2}\tilde{J}_2(p)$

<0.977) and the numerical results indicate that for large p, C_2 is a constant,

$$C_2 \simeq 0.56 \pm 0.01 \tag{80}$$

This implies for large p that at T_c

$$\langle \log Z_p \rangle = \frac{L}{T} (\log K)^{1/2} - \frac{1}{2} \log p + \log C_2 \tag{81}$$

As for expression (76), we see that if $L \to \infty$ at fixed p, the finite-size correction is of order 1 as in the REM (64), whereas if $p \to \infty$ when $L \to \infty$ with p/L fixed (for example, $p/L = 1$ to obtain the directed polymer problem), one recovers a correction $-(T_c/2T) \log L$ similar to the finite-size correction for the KPP equation [see (42)].

In principle one could use (70) to study the neighborhood of T_c. The expression of the scaling form which describes the limit $T \to T_c$ and $L \to \infty$ becomes more complicated. Let us just mention what happens: If $L \to \infty$ with fixed p, one obtains as in the REM a scaling form which depends on the product $\sqrt{L}\,(T_c - T)$ [see (64)]. On the contrary, in the case $L \to \infty$ with p/L fixed ($p/L = 1$ for the polymer problem), one finds that the scaling form depends on the product $L(T_c - T)$. This result can be anticipated by comparing (76) and (81) and by noticing that the finite-size corrections become comparable when $L(T_c - T)$ is of order 1.

6. THE DISTRIBUTION OF THE ENERGY AMONG THE p STEPS OF THE GREM

In this section we shall examine how the energy of a typical configuration is distributed among the p steps of a GREM, As was discussed in Sections 2 and 5, if one takes p to be large, one expects to obtain a result valid for the mean field directed polymer problem. We shall again use this approach here.

It has been shown(20) that for the mean field directed polymer problem in the low-temperature phase, the probability distribution of the overlap between two walks consists of two delta functions: one at zero overlap and one at an overlap one. This means that two typical walks either spend only a vanishing fraction of their lengths together and then split, or that they remain together and only split a length l from the bottom of the tree, where $\lim_{L \to \infty} l/L = 0$. This implies that the energy of a typical walk must be concentrated in the first fraction of its length. Once two typical walks remain together for a finite fraction of their lengths, this then forces them to remain together to give an overlap of one, as only by following this route can the anomalously large energy on the initial portion of the walk be balanced to give a low overall energy. From the results obtained for the overlaps, one can therefore conclude that the initial fraction of a typical walk has an excess of energy compared with the energy expected for a walk of that length and that this excess of energy must increase with the system size N. In this section we shall show, using the p-step GREM, that for arbitrary p, and hence presumably for the mean field directed polymer, the excess of energy scales as $N^{1/2}$. The same result has been suggested for the polymer problem on a tree.(22)

To see how one can obtain information about energies a finite fraction along a path, it is useful to consider as an example a two-step GREM. Suppose that the energy distributions for this GREM are characterized by widths $(N\tilde{a}_1)^{1/2}$ and $(N\tilde{a}_2)^{1/2}$. Let us define a type of "partition function" involving two temperatures T_1 and T_2 by

$$\tilde{Z}(T_1, T_2) = \sum_{\mu} \exp\left[-\left(\frac{\varepsilon_1^{\{\mu\}}}{T_1} + \frac{\varepsilon_2^{\{\mu\}}}{T_2}\right)\right] \tag{82}$$

where $\varepsilon_i^{\{\mu\}}$ is the energy on the ith step. With this definition one regains the normal partition function by choosing $T = T_1 = T_2$, i.e.,

$$\tilde{Z}(T, T) = Z_2(T) \tag{83}$$

One can obtain the average energy on a particular step of the GREM as a derivative with respect to $1/T_i$:

$$\bar{\varepsilon}_i = -\frac{d \log \tilde{Z}(T_1, T_2)}{d(1/T_i)}, \qquad i = 1, 2 \tag{84}$$

where the overbar represents a thermal average. Choosing the energy distributions to have widths $(N\tilde{a}_1)^{1/2}$ and $(N\tilde{a}_2)^{1/2}$ and allowing two temperatures T_1 and T_2 is equivalent to fixing one temperature T and selecting the widths to be

$$(Na_i)^{1/2} = (N\tilde{a}_i)^{1/2} \frac{T}{T_i} \tag{85}$$

Hence, one can rewrite (84) as

$$\bar{\varepsilon}_i = -\frac{d \log Z(T)}{d\sqrt{a_i}} \frac{d\sqrt{a_i}}{d(1/T_i)} = T\sqrt{a_i} \frac{d}{d\sqrt{a_i}} [-\log Z(T)] \tag{86}$$

Clearly this argument can be extended to allow one to determine the average energy of a given fraction of the length of walks for a GREM of an arbitrary number of steps. Here we shall only consider the simultaneous freezing case of a p-step GREM with $A_i = A + z_i N^{-1/2}$, with z_i of order one, in the low-temperature phase (Section 3.2). One then has that [see (10)]

$$\sqrt{a_i} = \frac{2(\log \alpha_i)^{1/2}}{A} - \frac{2(\log \alpha_i)^{1/2}}{A^2 N^{1/2}} z_i \tag{87}$$

So, the average energy of the first j steps of the GREM, $\langle \overline{E(j)} \rangle$ $(j \leqslant p)$, is given by

$$\langle \overline{E(j)} \rangle = A N^{1/2} \sum_{i=1}^{j} \frac{\partial}{\partial z_i} \langle T \log Z_p \rangle \tag{88}$$

Having obtained a formula from which to derive the average energy of the first j steps of the p-step GREM from the average free energy, we can now use the results of Section 3.2 to obtain an explicit expression for $\langle \overline{E(j)} \rangle$. For the case under consideration, we have shown that $\langle \log Z_p \rangle$ is given by (59) and (60). Taking the derivatives with respect to z_i in (59) and (60), one obtains, to order $N^{1/2}$,

$$\begin{aligned} \langle \overline{E(j)} \rangle &= -N \sum_{i=1}^{j} (a_i \log \alpha_i)^{1/2} + N^{1/2} \sum_{i=1}^{j} \frac{1}{J_1(p-1)} \frac{\partial}{\partial z_i} J_1(p-1) \\ &= -N \sum_{i=1}^{j} (a_i \log \alpha_i)^{1/2} + N^{1/2} \frac{I(j, p-1)}{J_1(p-1)} \end{aligned} \tag{89}$$

where

$$\begin{aligned} I(j, p-1) = {} & \int_0^{\infty} \frac{d\psi_1}{(\pi a_1)^{1/2}} \cdots \int_0^{\infty} \frac{d\psi_{p-1}}{(\pi a_{p-1})^{1/2}} \psi_j \\ & \times \exp\Bigg[-(z_p - z_{p-1}) \psi_{p-1} \\ & - (z_{p-1} - z_{p-2}) \psi_{p-2} \cdots - (z_2 - z_1) \psi_1 \\ & - \frac{\psi_1^2}{a_1} - \frac{(\psi_2 - \psi_1)^2}{a_2} \cdots - \frac{(\psi_{p-2} - \psi_{p-1})^2}{a_{p-1}} - \frac{\psi_{p-1}^2}{a_p} \Bigg] \end{aligned} \tag{90}$$

One can therefore see that there is an excess of energy on the initial portion of a typical walk and that this excess increases with the system size as $N^{1/2}$.

If we now take the limit of large p, with the $\{a_i\}$ and $\{\alpha_i\}$ given by (26), we see that we need to evaluate the integrals $J_1(p-1)$ and $I(j, p-1)$

in the case where all the $z_i = 0$ and where all the a_i are equal. Therefore, if in this case we denote j/p by

$$\lambda = j/p \tag{91}$$

we have

$$\langle \overline{E(j)} \rangle = -L\lambda(\log K)^{1/2} + \left(\frac{L}{p}\right)^{1/2} \frac{\tilde{I}(j, p-1)}{\tilde{J}_1(p-1)} \tag{92}$$

where $\tilde{J}_1(p-1)$ and $\tilde{I}(j, p-1)$ are defined by (73) and by

$$\begin{aligned}\tilde{I}(j, p-1) = \int_0^\infty \frac{d\psi_1}{\sqrt{\pi}} \cdots \int_0^\infty \frac{d\psi_{p-1}}{\sqrt{\pi}} \psi_j \\ \times \exp[-\psi_1^2 - (\psi_2 - \psi_1)^2 \cdots - (\psi_{p-2} - \psi_{p-1})^2 - \psi_{p-1}^2]\end{aligned} \tag{93}$$

We already know the large-p behavior of $\tilde{J}_1(p-1)$, (74)–(75). Thus, we only need to study the integral $\tilde{I}(j, p-1)$ for large p. It turns out that one can express $\tilde{I}(j, p-1)$ in terms of the $\tilde{J}_1(q)$: For $2 \leqslant j \leqslant p-2$, one can show that

$$\begin{aligned}\tilde{I}(j, p-1) = \frac{1}{2\sqrt{\pi}} \tilde{J}_1(0) \tilde{J}_1(p-2) + \frac{j}{4\sqrt{\pi}} \sum_{i=j+1}^{p-j-1} \tilde{J}_1(i-1) \tilde{J}_1(p-i-1) \\ + \frac{1}{2\sqrt{\pi}} \sum_{i=2}^{j} i\tilde{J}_1(i-1) \tilde{J}_1(p-i-1)\end{aligned} \tag{94}$$

where

$$\tilde{J}_1(0) = 1 \qquad \text{and} \qquad \tilde{J}_1(1) = \int_0^\infty \frac{dx\, e^{-2x^2}}{\sqrt{\pi}} = \frac{1}{2\sqrt{2}} \tag{95}$$

We shall not give a detailed derivation of (94) here. Let us mention the main steps which lead to it. First, it is easy to check that

$$\begin{aligned}2\tilde{I}(j, p-1) - \tilde{I}(j+1, p-1) - \tilde{I}(j-1, p-1) \\ = \frac{1}{2\sqrt{\pi}} \tilde{J}_1(j-1) \tilde{J}_1(p-j-1)\end{aligned} \tag{96}$$

So $\tilde{I}(j, p-1)$ can be expressed in terms of the $\tilde{J}_1$ and of $\tilde{I}(1, p-1)$ and $\tilde{I}(2, p-1)$.

The second step is to use the symmetry (93)

$$\tilde{I}(j, p-1) = \tilde{I}(p-j, p-1) \tag{97}$$

$\tilde{I}(1, p-1)$ can be determined in terms of $\tilde{J}_1$-type integrals by (97) and by integrating (93) by parts $p-1$ times. Also using (96) and (97), one can obtain $\tilde{I}(2, p-1)$. Doing so, one obtains expression (94).

So we see that the knowledge of the large-p behavior of $\tilde{J}_1(p-1)$ also gives the asymptotic form of $\tilde{I}(j, p-1)$,

$$\begin{aligned}\tilde{I}(j, p-1) &= \frac{C_1^2}{2p\sqrt{\pi}}\left[\lambda\int_\lambda^{1/2}\frac{dx}{x^{3/2}(1-x)^{3/2}} + \int_0^\lambda \frac{dx}{x^{1/2}(1-x)^{3/2}}\right] \\ &= \frac{2C_1^2}{p\sqrt{\pi}}\lambda^{1/2}(1-\lambda)^{1/2} \end{aligned} \tag{98}$$

Therefore (92) becomes

$$\langle\overline{E(j)}\rangle = -L\lambda(\log K)^{1/2} + \frac{2C_1}{\sqrt{\pi}}\lambda^{1/2}(1-\lambda)^{1/2}\sqrt{L} \tag{99}$$

So we obtain for the polymer problem that the excess energy of the first λL steps is proportional to $L^{1/2}$ and the coefficient is a semicircular law.

In principle one could imagine using procedures other than (26) to obtain the polymer problem in the large-p limit. We tried other procedures where the $\{a_i\}$ and $\{\log \alpha_i\}$ become small in a more complicated way than in (26). For these more complicated procedures, we did not succeed in calculating the leading correction to the energy and so we could not test whether the result given in (99) is independent of the procedure used.

7. CONCLUSION

In the present paper we first obtained the leading finite-size corrections for a GREM with an arbitrary number of steps p, (51), (60). We saw that these finite-size corrections signal the phase transition by becoming singular when $T \to T_c$. We also obtained the scaling forms (63), (70) for these corrections which are valid when $N \to \infty$ and $T \to T_c$. By making the number of steps very large, we obtained the same finite-size corrections as those predicted from the traveling wave approach based on the study of the KPP equation. Lastly, we saw that from the knowledge of the finite-size corrections one can calculate how the energy of a typical path is distributed along the path.

A possible extension of the present work could be the study of the finite-size corrections in the high-temperature phase. In our approach, based on the traveling wave equation (44), we neglected terms exponentially small in N which are probably of the same order as the finite-size corrections in the high-temperature phase (at least for the REM[(3)]).

Another extension could be to calculate other quantities such as the magnetic susceptibility(10) or the overlaps(10,20) to see how the finite-size effects affect the transition. Lastly, one could try to extend our results to distributions of energies more general than the Gaussian (19). We believe that the logarithmic corrections ($-1/2 \log L$ and $-3T_c/2T \log L$) given in (42) and (43) remain valid for a large class of distributions $\rho(\varepsilon)$ [probably as long as the distribution $\rho(\varepsilon)$ decays faster than any exponential when $|\varepsilon| \to \infty$].

Although the results obtained in the present paper are limited to the finite-size corrections for the GREM and the mean field limit of directed polymers in a random medium, they might be a useful step toward a better understanding of two problems: the finite-size corrections in spin glasses and velocity selection in traveling wave problems.

In spin glasses, at present, the replica approach developed by Parisi is considered to be the correct solution of the mean field spin glass. The numerical work(27,28) done on the Sherrington–Kirkpatrick model to test this mean field theory seems to agree with the Parisi solution, but the absence of theoretical predictions for the finite-size corrections always makes the comparison difficult. Even in the case of the REM, the finite-size corrections, which are known,(3) have not yet been obtained by the replica approach. It would be very interesting to understand how the finite-size corrections obtained in this paper for the REM, the GREM, or the directed polymer could be recovered using replicas. This would certainly be a useful step toward connecting the replica calculations and the numerical data have been obtained for a large variety of systems (spin glasses, neural networks, optimization problems, etc.) where it is always difficult to compare the results of the replica calculations done in the thermodynamic limit with those of the numerical simulations.

Traveling wave equations and the problem of front propagation appear in many physical systems(29,30) (for example, flow or instabilities). One often has the problem of locating a front which moves from a stable to an unstable region [for the traveling wave equations (15), (29), (35) the stable region is where $G = 0$, and the unstable region corresponds to $G = 1$]. One knows that in general there exists a one-parameter family of shapes for the front, indexed by their velocity. The problem is to predict the velocity, the shape, and the position of the front, given the traveling wave equation and the initial condition. This is a difficult problem with very few exactly soluble examples. In the present paper, we had to study a traveling wave equation and, by using the fact that N was large, we were able to calculate the position of the front step by step (Appendix A). For the GREM, the low-temperature phase always corresponds to the fact that if the initial condition $G_0(x)$ decays too quickly at infinity, the front ends up moving at the minimal velocity. It would be interesting to see whether our calculations could be extended to a larger class of traveling wave equations, in particular to cases where the minimal velocity solution is never accessible.(30,31)

Another possible extension of the present work would be to consider traveling wave equations with coefficients which vary with space, since for the GREM, the a_i and the α_i are *a priori* arbitrary functions of i.

APPENDIX A

In this Appendix, we are going to show that when one iterates (15) p times, the function $G_p(x)$ always has the form

$$G_p(x) \simeq \exp(-e^{-A_1(x-D)}) \tag{A1}$$

in the range of values of x where $G_p(x)$ is noticeably different from zero or one. In other words, if D is the value of x such that

$$G_p(D) = 1/e \tag{A2}$$

one expects (A1) to be valid in the whole range where $(x-D)$ is of order one. The main goal of this Appendix is to describe a recursion method which allows one to compute the constant D.

It turns out that one has to treat the two cases that we consider in Sections 3 and 4, the gradual freezing case $A_1 < A_2 < \cdots < A_p$ and the simultaneous freezing case $A_1 \simeq A_2 \simeq \cdots \simeq A_p$, in slightly different ways. The constants A_i are defined in (10),

$$A_i = 2\left(\frac{\log \alpha_i}{a_i}\right)^{1/2} \tag{A3}$$

First case: $A_1 < A_2 < \cdots < A_p$

If we assume that $G_i(x)$ has the form

$$G_i(x) = \exp[-e^{-A_{p+1-i}(x-D_i)}] \tag{A4}$$

we are going to see that $G_{i+1}(x)$ has a similar form and we shall relate the new constant D_{i+1} to D_i. Using the recursion (44) for G_i

$$G_{i+1}(x) = \exp\left[-(N\pi a_{p-i})^{-1/2}\,\alpha_{p-i}^{N} \int d\varepsilon[1 - G_i(x+\varepsilon)] \exp(-\varepsilon^2/Na_{p-i})\right] \tag{A5}$$

one can show that, if $G_i(x)$ is given by (A4), the integral over ε is dominated by the range

$$\varepsilon = -N(a_{p-i}\log\alpha_{p-i})^{1/2} + \frac{1}{2A_{p-i}}\log(4\pi N\log\alpha_{p-i}) + \varphi \tag{A6}$$

where φ is of order one. This integral over ε in (A5) can be done and by neglecting terms of order $N^{-1/2}$, one obtains that $G_{i+1}(x)$ has the same form as in (A4),

$$G_{i+1}(x) = \exp[-e^{-A_{p-i}(x-D_{i+1})}] \tag{A7}$$

with the constant D_{i+1} given by

$$\begin{aligned} D_{i+1} = D_i + N(a_{p-i}\log\alpha_{p-i})^{1/2} - \frac{1}{2A_{p-i}}\log(4\pi N\log\alpha_{p-i}) \\ + \frac{1}{A_{p-i}}\log\left[\Gamma\left(1-\frac{A_{p-i}}{A_{p+1-i}}\right)\right] \end{aligned} \tag{A8}$$

So, to determine the constant $D = D_p$ in (A1) one needs to know D_1 and to iterate (A8) $p-1$ times. The calculation of D_1 can be done by repeating almost the same procedure which related D_i to D_{i+1}. One knows from (16) that $G_0(x)$ has also the form (A1):

$$G_0(x) = \exp(-e^{-x/T}) \tag{A9}$$

Then, by using (A4) and (A5), one finds that

$$D_1 = N(a_p\log\alpha_p)^{1/2} - \frac{1}{2A_p}(4\pi N\log\alpha_p) + \tilde{D}_1 \tag{A10}$$

where $\tilde{D}_1$ is given to leading order by

$$\tilde{D}_1 = \frac{1}{A_p}\log\left\{\int_{-\infty}^{+\infty} d\varphi[1-\exp(-e^{-\varphi/T})]\,A_p\exp\left(A_p\varphi - \frac{\varphi^2}{Na_p}\right)\right\} \tag{A11}$$

This leads one to distinguish between two subcases:

(i) If $T < 1/A_p$, then the term φ^2/Na_p in (A11) can be neglected and (A11) becomes

$$\tilde{D}_1 = \frac{1}{A_p}\log[\Gamma(1-TA_p)] \tag{A12}$$

(ii) If $T \simeq 1/A_p$, and more precisely if

$$\frac{1}{T} - A_p = O\left(\frac{1}{\sqrt{N}}\right) \tag{A13}$$

the integral in (A11) is dominated by the range $\varphi > 0$ and $\varphi = O(\sqrt{N})$. This leads to

$$\tilde{D}_1 = \frac{1}{A_p} \log \left\{ A_p (N a_p)^{1/2} \exp\left[\frac{N a_p}{4}\left(A_p - \frac{1}{T}\right)^2\right] \int_{(Na_p)^{1/2}(1/T - A_p)/2}^{\infty} \exp(-t^2)\, dt \right\} \tag{A14}$$

We see that (A1), (A8), (A10), and (A12) lead to (49), whereas (A1), (A8), (A10), and (A14) give (61).

Second Case: $A_1 \simeq A_2 \simeq \cdots \simeq A_p$

We have already seen in the previous case that for $1/T$ close to A_p, one had to be more careful about the range of values of φ [defined in (A6)] which contribute.

When the differences $A_i - A_{i+1}$ are small,

$$A_i - A_{i+1} = O\left(\frac{1}{\sqrt{N}}\right) \tag{A15}$$

the same thing happens at each iteration and one needs to keep more information about $G_i(x)$.

If we consider that the A_i are given by (65)

$$A_i = A + z_i/\sqrt{N} \tag{A16}$$

we have to replace (A4) by

$$G_i(x) = \exp\left[-F_i\left(\frac{x - D_i}{\sqrt{N}}\right) e^{-A(x - D_i)}\right] \tag{A17}$$

Since $F_i(0)$ has no reason to remain equal to one, D_i is no longer the value of x where $G_i(x)$ equals $1/e$, but is some value x where $G_i(x)$ is noticeably different from one and zero. As in the previous case, we use recursion (A5) to relate D_i and F_i to D_{i+1} and F_{i+1}. For large N one obtains

$$D_{i+1} = D_i + N(a_{p-i} \log \alpha_{p-i})^{1/2} \tag{A18}$$

$$F_{i+1}(t) = \frac{1}{(\pi a_{p-i})^{1/2}} \int_0^{\infty} du\, F_i(u) \exp\left[-\frac{(u-t)^2}{a_{p-i}} + z_{p-i}(t-u)\right] \tag{A19}$$

So, to calculate $G_p(x)$, one needs to know D_1 and $F_1(t)$ and to iterate (A18) and (A19). Then the constant D in (A1) will be given by

$$D = D_p + \frac{1}{A} \log F_p(0) \tag{A20}$$

As in the previous case, one needs to distinguish between two subcases to calculate $F_1(t)$.

(i) If $A < 1/T$, from (A5) and the initial condition

$$G_0(x) = \exp(-e^{-x/T}) \tag{A21}$$

one gets that

$$D_1 = N(a_p \log \alpha_p)^{1/2} \tag{A22}$$

and by making the change of variable

$$\varepsilon = -N(a_p \log \alpha_p)^{1/2} - (x - D_1) + \varphi \tag{A23}$$

in (A5), we see that

$$\begin{aligned} G_1(x) = \exp\Bigg[& -\exp[-A_p(x - D_1)] \frac{1}{(\pi N a_p)^{1/2}} \\ & \times \int d\varphi \exp\left[A_p \varphi - \frac{(\varphi + D_1 - x)^2}{N a_p} \right] \\ & \times \left\{ 1 - \exp\left[-\exp\left(-\frac{\varphi}{T} \right) \right] \right\} \Bigg] \end{aligned} \tag{A24}$$

If the temperature T is noticeably lower than $1/A$, i.e., if $1/T - A \gg N^{-1/2}$, one can simplify (A24) and show $G_1(x)$ has the form (A17) with D_1 given by (A22) and F_1 given by

$$F_1(t) = \frac{1}{A_p(\pi N a_p)^{1/2}} \exp\left(-\frac{t^2}{a_p} - z_p t \right) \Gamma(1 - A_p T) \tag{A25}$$

Then, using (A18), (A19), (A22), and (A25), one easily obtains the results (57)–(59).

(ii) When $A \simeq 1/T$ the calculation remains identical up to (A24). Then if one defines z_{p+1} by

$$\frac{1}{T} = A + z_{p+1} N^{-1/2} \tag{A26}$$

one can show that the integral in (A24) is dominated by the range where φ is of order $\sqrt{N}$. Therefore, one obtains

$$F_1(t)=\frac{1}{(\pi a_p)^{1/2}}\int_0^\infty d\psi \exp\left(-\frac{\psi^2}{a_p}+\frac{2t\psi}{a_p}-\frac{t^2}{a_p}-z_{p+1}\psi\right) \tag{A27}$$

Then, using (A18), (A19), (A22), and (A27), one arrives at (67)–(69).

APPENDIX B

In this Appendix we present the result of a numerical simulation to estimate the constants C_1 and C_2 in (74) and (79) and we derive bounds on the integrals $\tilde{J}_1(p-1)$ and $\tilde{J}_2(p)$ defined in (73) and (78).

The integrals $\tilde{J}_1(p-1)$ and $\tilde{J}_2(p)$ are given by

$$\begin{aligned}\tilde{J}_1(p-1)=&\int_0^\infty \frac{d\psi_1}{\sqrt{\pi}}\int_0^\infty \frac{d\psi_2}{\sqrt{\pi}}\cdots\int_0^\infty \frac{d\psi_{p-1}}{\sqrt{\pi}}\\ &\times\exp[-\psi_1^2-(\psi_2-\psi_1)^2\cdots-(\psi_{p-1}-\psi_{p-2})^2-\psi_{p-1}^2]\end{aligned} \tag{B1}$$

and

$$\begin{aligned}\tilde{J}_2(p)=&\int_0^\infty \frac{d\psi_1}{\sqrt{\pi}}\int_0^\infty \frac{d\psi_2}{\sqrt{\pi}}\cdots\int_0^\infty \frac{d\psi_p}{\sqrt{\pi}}\\ &\times\exp[-\psi_1^2-(\psi_2-\psi_1)^2\cdots-(\psi_p-\psi_{p-1})^2]\end{aligned} \tag{B2}$$

These integrals can be interpreted in the following way. Consider a Brownian particle in one dimension, which starts at time $t=0$ at $x=0$ and returns to $x=0$ at time $t=p$. Then $\tilde{J}_1(p-1)$ is the probability that the position $x(t)$ of this Brownian particle will take a positive value at all integer times $1\leqslant t\leqslant p-1$. The interpretation of $\tilde{J}_2(p)$ is even simpler. $\tilde{J}_2(p)$ is the probability that the position of a Brownian particle starting at $x=0$ at time $t=0$ will be positive at all integer times $1\leqslant t\leqslant p$.

With these interpretations it is easy to perform a numerical simulation of these integrals. The results obtained by estimating $\tilde{J}_1(p-1)$ and $\tilde{J}_2(p)$ in a Monte Carlo simulation of 5×10^7 Brownian particles for $1\leqslant p\leqslant 400$ are shown in Figs. 3a and 3b. The results clearly indicate that for large p

$$\tilde{J}_1(p-1)\simeq\frac{C_1}{p^{3/2}} \tag{B3}$$

$$\tilde{J}_2(p)\simeq\frac{C_2}{p^{1/2}} \tag{B4}$$

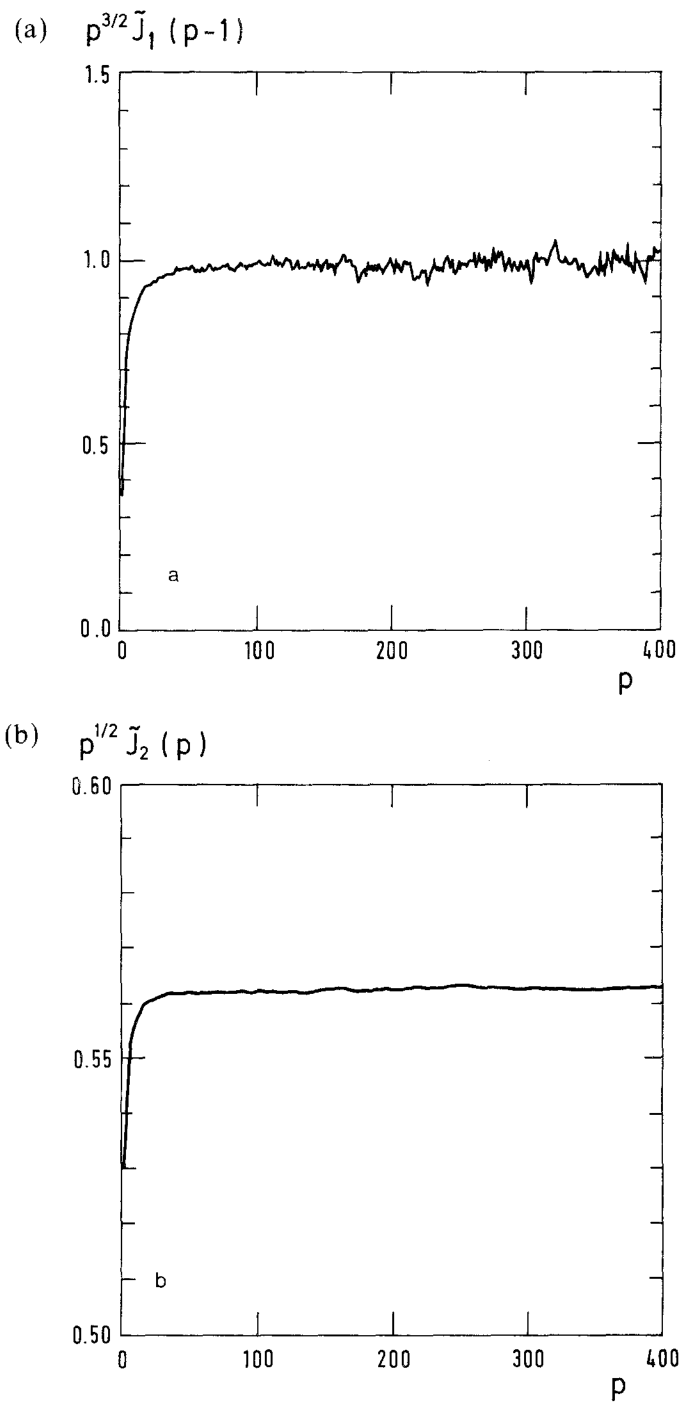

Fig. 3. (a) The result of a Monte Carlo calculation of the product $p^{3/2}\tilde{J}_1(p-1)$ versus p, where the integral $\tilde{J}_1(p-1)$ is defined in (B1). The statistics become worse as p increases because the number of surviving Brownian particles decreases with p. This leads to the estimate (B5). (b) As in part (a), for the product $p^{1/2}\tilde{J}_2(p)$ leading to the estimate (B6). The statistics are better because in this case all particles on the positive axis contribute, whereas in the case of part (a) only those close enough to 0 contribute.

with

$$C_1 = 1.00 \pm 0.05 \tag{B5}$$

$$C_2 = 0.56 \pm 0.01 \tag{B6}$$

We were not able to find an analytic expression for the constants C_1 and C_2. However, we can prove that the power law dependences, (B3) and (B4), are indeed correct (notice that it is only these power laws which are used in Section 5), by obtaining the following bounds for large p:

$$0.318 \simeq \frac{1}{\pi} < p^{3/2} \tilde{J}_1(p-1) < \frac{2A + A^2 \sqrt{\pi}}{2\sqrt{\pi}\,(1 - e^{-A^2})} \simeq 2.36 \tag{B7}$$

$$0.318 \simeq \frac{1}{\pi} < p^{1/2} \tilde{J}_2(p) < \frac{A}{\sqrt{\pi}\,(1 - e^{-A^2})} \simeq 0.977 \tag{B8}$$

for any constant A such that

$$A \geqslant (8/\pi)^{1/2} \tag{B9}$$

To obtain these bounds, it is convenient to introduce the function $\phi_p(x)$, defined by the following recursion:

$$\phi_{p+1}(x) = \int_0^\infty \frac{dt}{\sqrt{\pi}}\, e^{-(x-t)^2} \phi_p(t) \tag{B10}$$

with

$$\phi_0(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \tag{B11}$$

Clearly one has

$$\tilde{J}_1(p-1) = \int_0^\infty e^{-x^2} \phi_{p-2}(x)\, dx \tag{B12}$$

and

$$\tilde{J}_2(p) = \int_0^\infty \phi_{p-1}(x)\, dx \tag{B13}$$

Therefore, to obtain $\tilde{J}_1(p-1)$ and $\tilde{J}_2(p)$ for large p one just needs to calculate the behavior of $\phi_p(x)$.

We were unable to find an explicit analytic expression for $\phi_p(x)$ valid for large p. However, one can obtain bounds on $\phi_p(x)$. It turns out that one can prove that, for all positive integers p,

$$L_p(x) \leqslant \phi_p(x) \leqslant U_p(x) \tag{B14}$$

where

$$L_p(x)=\int_0^\infty dy\,\frac{e^{-y^2}}{\sqrt{\pi}}\left[\frac{e^{-(x-y)^2/p}-e^{-(x+y)^2/p}}{(\pi p)^{1/2}}\right] \tag{B15}$$

and

$$U_p(x)=\frac{1}{1-e^{-A^2}}\left[\frac{e^{-x^2/(p+1)}-e^{-(x+A)^2/(p+1)}}{[\pi(p+1)]^{1/2}}\right] \tag{B16}$$

for any constant A satisfying (B9). The proof of (B14) can be done by recursion. One first proves that (B14) is true for $p=0$ or $p=1$. Then one shows that if (B14) is satisfied for p, it remains true for $p+1$. The first step ($p=0$ for the upper bound and $p=1$ for the lower bound) is very easy. One can easily check that for $x\geqslant 0$ [see (B11), (B15), and (B16)]

$$L_1(x)\leqslant\phi_1(x)\qquad\text{and}\qquad\phi_0(x)\leqslant U_0(x) \tag{B17}$$

Then one needs to show that (B14) remains true on recursion. For the lower bound it is easy, since one can check that $L_p(x)$, given by (B15), satisfies the following recursion:

$$L_p(x)=\int_0^\infty dt\,L_{p-1}(t)\left[\frac{e^{-(x-t)^2}-e^{-(x+t)^2}}{\sqrt{\pi}}\right] \tag{B18}$$

Then, by comparing (B10) and (B18), it is clear that if $L_1(x)\leqslant\phi_1(x)$ [see (B17)], this inequality remains satisfied for all p. The fact that $L_p(x)$ gives a lower bound on $\phi_p(x)$ can be understood without any calculation. $\phi_p(x)$ is the probability that a Brownian particle, in one dimension, which starts at time $t=0$ at $x=0$, will reach the point x at time $t=p$ with the constraint that its displacement must be positive at all integer times $1\leqslant t\leqslant p$. The $L_p(x)$ is the same probability with a different constraint: the particle has to have a positive displacement at all real times $0<t\leqslant p$. Clearly this latter constraint is stronger and therefore $L_p(x)\leqslant\phi_p(x)$.

Let us now turn to the derivation of the upper bound. The linear operator (B10) transforms a positive function into a positive function. If one chooses the difference $U_p(x)-\phi_p(x)$ as a positive function, one knows that the transformation of $U_p(x)$ by the linear operator (B10) will be an upper bound on $\phi_{p+1}(x)$. Therefore, to prove that $U_{p+1}(x)$ is an upper bound on $\phi_{p+1}(x)$, it is sufficient to show that

$$\int_0^\infty dt\,U_p(x)\frac{e^{-(x-t)^2}}{\sqrt{\pi}}\leqslant U_{p+1}(x) \tag{B19}$$

By rewriting $U_{p+1}(x)$, given by (B16), as

$$U_{p+1}(x)=\int_{-\infty}^{+\infty} dt\, U_p(t)\,\frac{e^{-(x-t)^2}}{\sqrt{\pi}} \tag{B20}$$

we see that to prove (B19) one only needs to show that

$$B_p(x)=\int_{-\infty}^{0} dt\, U_p(t)\,\frac{e^{-(x-t)^2}}{\sqrt{\pi}}\geqslant 0 \tag{B21}$$

To avoid too long a discussion, let us limit our proof of (B21) to the main steps. First, by separating the integral over t in (B21) into two parts $(-\infty\leqslant t\leqslant -A/2)$ and $(-A/2\leqslant t\leqslant 0)$, one can show that

$$B_p(x)\geqslant\frac{Ke^{-x^2-Ax}}{[\pi(p+1)]^{1/2}\,(1-e^{-A^2})} \tag{B22}$$

where

$$K=\int_0^{\infty} dt\, e^{-t^2}\left(e^{-t^2/(p+1)}-e^{-(t-A)^2/(p+1)}\right) \tag{B23}$$

So, it only remains to prove that the constant K in (B23) is positive if A satisfies (B9). One can easily demonstrate that (B23) is equivalent to

$$Ke^{A^2/(p+2)}\,\frac{2(p+2)^{1/2}}{[\pi(p+1)]^{1/2}}=\left(e^{A^2/(p+2)}-1\right)-\frac{2}{\sqrt{\pi}}\int_0^{A/[(p+1)(p+2)]^{1/2}} du\, e^{-u^2} \tag{B24}$$

The right-hand side of (B24) is certainly larger than

$$\frac{A^2}{p+2}-\frac{2A}{[\pi(p+1)(p+2)]^{1/2}} \tag{B25}$$

which is positive for all $p\geqslant 0$ if

$$A\geqslant\left(\frac{8}{\pi}\right)^{1/2} \tag{B26}$$

The condition (B26) is therefore a sufficient condition for the upper bound (B14) to be valid for all integer $p\geqslant 0$.

Having established (B14), it is easy to obtain the bounds (B7) and (B8) from the expressions (B12), (B13), (B15), and (B16).

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